

On Vertex Rankings of Graphs and its Relatives

Ilan Karpas*

Ofer Neiman†

Shakhar Smorodinsky‡

Abstract

A vertex ranking of a graph is an assignment of ranks (or colors) to the vertices of the graph, in such a way that any simple path connecting two vertices of equal rank, must contain a vertex of a higher rank. In this paper we study a relaxation of this notion, in which the requirement above should only hold for paths of some bounded length l for some fixed l . For instance, already the case $l = 2$ exhibit quite a different behavior than proper coloring. We prove upper and lower bounds on the minimum number of ranks required for several graph families, such as trees, planar graphs, graphs excluding a fixed minor and degenerate graphs.

1 Introduction

A *vertex-ranking* of a graph $G = (V, E)$ with k colors, is a k -coloring $c : V \rightarrow [k]$ such that for any $u, v \in V$, if $c(u) = c(v)$ then any simple path between u, v contains a vertex w with $c(w) > c(u)$ (note that this implies that the coloring is proper). The minimum k such that G admits a vertex-ranking with k colors is called the *vertex ranking chromatic number* (abbreviated vr-number), and is denoted by $\chi_{\text{vr}}(G)$. In this paper we introduce a natural relaxation of vertex-ranking to the case where the above requirement holds only for paths of length bounded by l . In particular, we focus on the special case $l = 2$.

Definition 1.1. Let $G = (V, E)$ be a simple graph. A *unique-superior* coloring (abbreviated *us-coloring*) of G with k colors is a function $c : V \rightarrow [k]$, such that c is a proper coloring, and furthermore, for every length two simple path (u, v, w) , if $c(u) = c(w)$ then $c(v) > c(u)$. The minimum k for which a graph G admits a us-coloring with k colors is called the *unique-superior chromatic number* of G (abbreviated *us-number*), and is denoted by $\chi_{\text{us}}(G)$.

More generally, in an l -vertex ranking of G with k colors, every simple path of length at most l between two vertices of the same color contains a vertex of higher color. The minimum k for which G admits an l -vertex ranking is called the *l -vr-number* of G , and is denoted by $\chi_{l\text{-vr}}(G)$.

Note that trivially we have:

$$\chi(G) \leq \chi_{\text{us}}(G) \leq \chi_{\text{vr}}(G).$$

The vr-number is a well studied notion, with both theoretical and practical applications. It has been used for VLSI lay-out design, Cholesky matrix factorization and scheduling assembly

*Mathematics department, Ben-Gurion University, Be'er Sheva 84105, Israel. karpasi@math.bgu.ac.il. Work on this paper was conducted while the author was an M.Sc student under the supervision of Shakhar Smorodinsky.

†Computer Science department, Ben-Gurion University, Be'er Sheva 84105, Israel. neimano@cs.bgu.ac.il. Supported in part by ISF grant No. (523/12) and by the European Unions Seventh Framework Programme (FP7/2007-2013) under grant agreement n°303809.

‡Mathematics department, Ben-Gurion University, Be'er Sheva 84105, Israel. shakhar@math.bgu.ac.il, work on this paper by Shakhar Smorodinsky was supported by Grant 1136/12 from the Israel Science Foundation.

problems (see, e.g., [IRV88, Lei80, Liu90]). Recently, Even and Smorodinsky [ES11] showed a strong connection between vertex ranking of graphs and some online hitting set problems on graphs. Efficient polynomial algorithms to determine the vr-number of graphs have been found for several families of graphs: Schäffer showed this for trees [Sch88] and Deugun et al. for permutation graphs [DKKM94]. For other families of graphs, such as chordal graphs, bipartite and co-bipartite graphs, to name a few, the problem was shown to be NP-hard [DN06, BDJ⁺98].

As mentioned above, for every graph G $\chi(G) \leq \chi_{\text{us}}(G) \leq \chi_{\text{vr}}(G)$. However, these parameters can be arbitrarily far apart already for bipartite graphs. Indeed, consider for example the family of trees. It is a well known fact that every tree T is bipartite i.e., $\chi(T) = 2$. However, it is not a difficult exercise to show that for every path P of length n we have $\chi_{\text{vr}}(P) = \Omega(\log n)$. See e.g., [KMS95, Smo12]. More generally, consider the family of planar graphs (a superset of the family of trees). By the famous Four-Color Theorem [AH77] we have $\chi(G) \leq 4$ for every planar graph G . Katchalski et al. [KMS95] proved that for any planar graph G with n vertices, $\chi_{\text{vr}}(G) = O(\sqrt{n})$. They also showed that this bound is best possible, i.e., that for any integer n there is a planar graph G with n vertices such that $\chi_{\text{vr}}(G) = \Omega(\sqrt{n})$. Their proof extends to any fixed minor-free family. That is, a family of graphs excluding a fixed subgraph H as a minor (see below for details). Another family of graphs for which the chromatic number and the vr-number can be far apart, is the family of 2-degenerate graphs (or more generally k -degenerate graphs). A graph is called d -degenerate if every subgraph has a vertex of degree at most d . It is well known and an easy fact to prove that a d -degenerate graph is always $d + 1$ -colorable. However, for any integer n there are 2-degenerate graphs G with n vertices such that $\chi_{\text{vr}}(G) = \Omega(n)$.

1.1 Related Notions of Coloring

The notion of vertex-ranking is in itself a special case of the more general notion of *unique-maximum* coloring of hypergraphs.¹ A *unique-maximum* coloring (abbreviated *um-coloring*) of a hypergraph H is a coloring of its vertices such that for every hyperedge $e \in E$, the highest color of a vertex of e is unique in e . Then, for a given graph $G = (V, E)$, a vertex ranking of G is exactly a um-coloring of the hypergraph $H = (V, E')$, where E' is the family of all subsets of vertices that form a simple path in G .

A slightly different variant of the notion of um-coloring is that of *conflict-free coloring*. This notion is a relaxation of um-coloring. In a conflict-free coloring of a hypergraph, the restriction is that for every hyperedge there is some color that occurs exactly once in that hyperedge. This coloring has been studied in many settings for the last decade, see the survey [Smo12] and the references therein.

There are other types of coloring that have been studied in the literature, which are more relaxed than us-coloring, yet stricter than proper coloring. Two such colorings are *acyclic coloring* and *star coloring*. An acyclic coloring of a graph G is a proper coloring, such that, for every cycle of G , at least three colors appear in the cycle. Equivalently, its a proper coloring such that every subgraph induced by the union of two color classes is acyclic. A star coloring of a graph G is a proper coloring, for which every path of length 4 in G uses at least 3 colors. Equivalently, it is a proper coloring such that every subgraph induced by the union of two color classes is a collection of stars. the star chromatic number and the acyclic chromatic number of G , $\chi_s(G)$ and $\chi_a(G)$, respectively, are defined in the obvious manner. It is easy to see that we have the following hierarchy: Let G be a graph, then $\chi(G) \leq \chi_a(G) \leq \chi_s(G) \leq \chi_{\text{us}}(G) \leq \chi_{\text{vr}}(G)$. Grünbaum [Gru73] showed that, for every planar graph G , $\chi_a(G) \leq 9$, and conjectured that $\chi(G) \leq 5$ for every planar

¹A hypergraph $H = (V, E)$, has vertices V and hyperedges $E \subseteq 2^V$.

graph, a conjecture which Borodin [Bor79] later proved (this bound is known to be tight). A few years later, Nešetřil and Ossona de Mendez [NOdM03] showed that the acyclic chromatic number is bounded for every minor-excluding family. Albertson et al. [ACK⁺04] proved that, for every graph G : $\chi_s(G) \leq 2\chi_a(G)^2 + \chi_a(G)$, which means that the star coloring chromatic number is also bounded for graphs excluding a fixed minor.

1.2 Our Results

We provide lower and upper bounds on the us-number and the l -vr-number of several graph families. For trees we show a tight bound of $\Theta(\log n / \log \log n)$ on the us-number, where n is the number of vertices. Thus already for trees the us-number can be far apart both from the chromatic number and the vr-number. For planar graphs, and more generally, graphs excluding a fixed minor, we show a nearly tight upper bound of $O(\log n)$ on the us-number and $O(l \cdot \log n)$ on the l -vr-number. For the family of d -degenerate graphs we show a lower bound of $\Omega(n^{1/3})$, and an upper bound of $O(\sqrt{n})$ on the us-number.

2 Preliminaries

All graphs mentioned in this paper are simple and undirected. For a graph $G = (V, E)$ we denote by $\Delta(G)$ (respectively, $\delta(G)$) its maximum (resp. minimum) degree. For a subset $V' \subseteq V$, $G[V']$ is the induced subgraph on V' , and $G \setminus V'$ is the graph induced on $V \setminus V'$. For a subgraph H of G , we may write $G \setminus H$ which means $G \setminus V(H)$. For a graph $G = (V, E)$, a *subdivision* of an edge $\{u, v\} \in E$ means adding a new vertex w on the edge which is connected only to u and to v . In other words, it is the graph $G' = (V \cup \{w\}, E \cup \{u, w\} \cup \{w, v\} \setminus \{u, v\})$.

Definition 2.1. A graph $G = (V, E)$ is called *d-degenerate* if for every $V' \subseteq V$, $\delta(G[V']) \leq d$.

Let \mathbb{G}_d denote the family of *d-degenerate graphs*. It is easy to see that $\chi(G) \leq d + 1$ for every $G \in \mathbb{G}_d$. The following propositions are well known, we include proofs for completeness.

Proposition 2.2. *Let $G = (V, E) \in \mathbb{G}_d$ be a graph with n vertices. Then there is an ordering of the vertices (v_1, \dots, v_n) , so that for every $1 \leq i \leq n$, the vertex v_i has at most d neighbors v_j with $j < i$.*

Proof. By induction: let $v \in V$ be a vertex of degree at most d . By the induction hypothesis, the vertices of $G \setminus v$ can be ordered (v_1, \dots, v_{n-1}) so that for all $1 \leq i \leq n-1$, v_i has at most d neighbors to its left. Put $v_n = v$ and obviously the order (v_1, \dots, v_n) satisfies the desired property. \square

Proposition 2.3. *Let $G = (V, E)$ be a graph, $|V| = n$, $|E| = m$, and let $U \subseteq V$ be a subset of vertices of degree at least d . Then $|U| \leq 2m/d$.*

Proof. By summing the degrees of the vertices of G , we have:

$$2m = \sum_{v \in V} d(v) \geq \sum_{v \in U} d(v) \geq d|U| ,$$

so that $|U| \leq 2m/d$. \square

The *square* of a graph $G = (V, E)$, denoted by $G^2 = (V, E')$, is the graph on the vertex set V with $\{u, v\} \in E'$ if and only if $d_G(u, v) \in \{1, 2\}$ where $d_G(u, v)$ denotes the length of a shortest path between u and v . Observe that a proper coloring of G^2 is a us-coloring of G (because there are no paths of length two whose end-points may get the same color).

3 The us-number of Trees

In this section we focus on the us-number of trees, and provide sharp asymptotic bounds.

Theorem 3.1. *For every tree T on n vertices,*

$$\chi_{us}(T) = O\left(\frac{\log n}{\log \log n}\right).$$

Theorem 3.2. *For any integer n there exists a tree T with n vertices such that $\chi_{us}(T) = \Omega\left(\frac{\log n}{\log \log n}\right)$*

3.1 Upper bound

Let $k > 3$ be some integer. It suffices to show that if a tree T has $\chi_{us}(T) = k$ then $|V(T)| \geq \Omega((k-2)!)$, because using standard bounds on $k!$ we have that $\log n = \Omega(k \log k)$. To this end, define for each $1 \leq i \leq k-1$ a boolean function p_i on the set of all rooted trees as follows: For a tree T with root r , $p_i(T, r) = 1$ if and only if for every two colors $i \leq l, m \leq k-1$, there is a us-coloring of T with $k-1$ colors, so that r has color l , and none of its children have color m . Because to have $p_i(T, r) = 1$ it must be that T admits a us-coloring with $k-1$ colors, we have the following:

Observation 1. *If $\chi_{us}(T) = k$ then $p_i(T, r) = 0$ for every i and every $r \in V(T)$.*

Define the function $f : [k-1] \rightarrow \mathbb{N}$ by

$$f(i) = \min\{n : \text{there exists a tree } T \text{ on } n \text{ vertices with root } r \in V(T) \text{ such that } p_i(T, r) = 0\}.$$

By [Observation 1](#), if $\chi_{us}(T) = k$ then $|V(T)| \geq f(k-1)$, thus proving the following Lemma will conclude the proof.

Lemma 3.3. $f(k-1) > (k-2)!$.

Proof. We show that for every $1 \leq i \leq k-3$,

$$f(i) \geq 1 + (k-i-1)f(i-1).$$

Let T be a tree with root u , such that $p_i(T, u) = 0$ and $|V(T)| = f(i)$. Denote by u_1, \dots, u_s the children of u , and by T_{u_j} the sub-tree of T rooted in u_j . Seeking contradiction, assume that $f(i) < 1 + (k-i-1)f(i-1)$. Then there are at most $k-i-2$ trees T_{u_j} for which $p_{i-1}(T_{u_j}, u_j) = 0$, since the number of vertices in each such tree is at least $f(i-1)$. By renaming we may assume that for $1 \leq j \leq s'$, $p_{i-1}(T_{u_j}, u_j) = 0$, and for $s'+1 \leq j \leq s$, $p_{i-1}(T_{u_j}, u_j) = 1$, where $s' \leq k-i-2$.

Fix any $i \leq l, m \leq k-1$. We demonstrate that there is a us-coloring c of T with at most $k-1$ colors, so that $c(u) = l$ and $c(u_j) \neq m$ for all $j \in [s]$. First, for each $j > s'$, as $p_{i-1}(T_{u_j}, u_j) = 1$ there is a coloring c_j of T_{u_j} such that $c_j(u_j) = i-1$ and $c_j(w) \neq l$ for every child w of u_j . As for $1 \leq j \leq s'$, we use the fact that T has the minimal number of vertices for which $p_i(T, u) = 0$, thus it must be that $p_i(T_{u_j}, u_j) = 1$. This implies that for every color d out of the color set $A = \{i, \dots, k-1\} \setminus \{l, m\}$ there exists a us-coloring c_j of T_{u_j} so that $c_j(u_j) = d$ and $c_j(w) \neq l$ for every child w of u_j . As $|A| = k-i-2$, we can choose a color $d(j)$ such that $d(j) \neq d(j')$ for all $s' < j < j' \leq s$. Finally, color the root u with color l . See [Figure 1](#) for an illustration. Then it can be verified that the obtained coloring c is a valid us-coloring of T satisfying the constraints imposed by the definition of $p_i(T, r)$. But this means that $p_i(T, u) = 1$, a contradiction. Using that recursive formula, we have $f(k-3) > (k-3)!f(1)$. One can check that $f(1) = k-2$, so $f(k-3) > (k-2)!$, and of course also $f(k-1) > (k-2)!$. \square

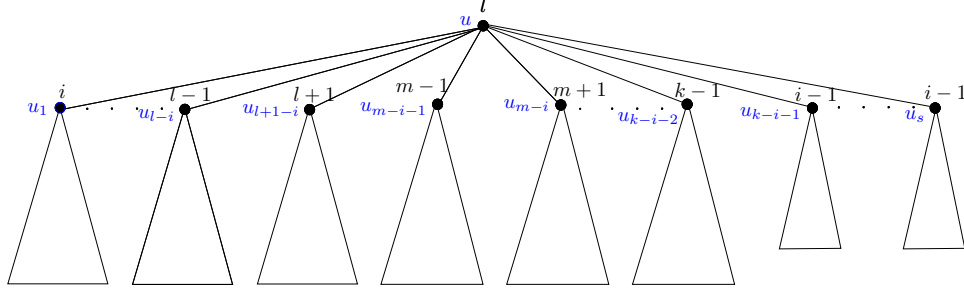


Figure 1: $s' = k - i - 2$, and $l \leq m$. Not shown in the figure are the grandchildren of u , all of which have colors different from l .

3.2 Lower bound

Next, we show that for every n , there exist a tree T on n vertices such that $\chi_{\text{us}}(T) = \Omega(\frac{\log n}{\log \log n})$. Let $T = T_k$ be the *complete k -ary tree with k levels*, i.e., a rooted tree of height $k - 1$, where each non-leaf has exactly k children, and all leaves have distance $k - 1$ from the root. Since $|V(T_k)| = \frac{k^k - 1}{k - 1}$ it follows that $k = \Omega(\frac{\log n}{\log \log n})$. The *level* of a vertex is its distance to the root, so the root is at level 0, its children are at level 1, and the leaves are at level $k - 1$. See Figure 2 for an illustration with $k = 3$. We shall prove that $\chi_{\text{us}}(T_k) = k = \Omega(\frac{\log n}{\log \log n})$.

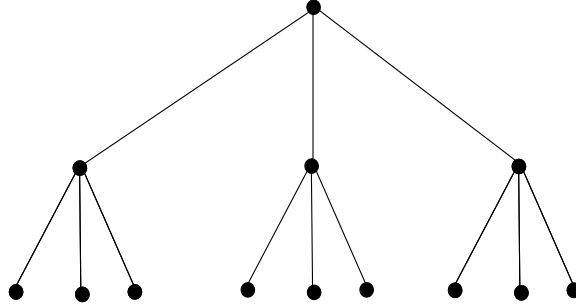


Figure 2: T_3 .

Lemma 3.4. *Let c be a us-coloring of T_k with k colors. Then for every $1 \leq i \leq k$, color i can be used only for vertices from the bottom i levels.*

Proof. The proof is by induction on i . The base case is for $i = 1$, and seeking contradiction, assume that $c(v) = 1$ and v is not a leaf. Note that v has exactly k children u_1, \dots, u_k . Since $c(v) = 1$ and the coloring is proper, then $c(u_l) > 1$, for all $l \in [k]$. Furthermore, $c(u_l) \neq c(u_m)$ for every $1 \leq l < m \leq k$, since v is a common neighbor of u_l and u_m with the lowest color. Hence, at least $k + 1$ colors are used, a contradiction.

The induction hypothesis is that the statement holds for all $j < i$. Seeking contradiction, assume that $c(v) = i$ for some vertex $v \in V(T_k)$ which is not in the bottom i levels. Denote by u_1, \dots, u_k the children of v . We know that $c(u_l) > i - 1$ for every $1 \leq l \leq k$ by the induction hypothesis, since u_l is not in the bottom $i - 1$ levels. Moreover, for every $l \in [k]$, since c is also a proper coloring then $c(u_l) \neq c(v) = i$, and we conclude that $c(u_l) > i$. This means that $c(u_l) \neq c(v)$ and $c(u_l) \neq c(u_m)$ for every $1 \leq l < m \leq k$. Again, this implies that $k + 1$ colors are used for v, u_1, \dots, u_k , a contradiction. This completes the induction step, proving the lemma. \square

Now we are ready to prove that $\chi_{\text{us}}(T_k) = k$. It can be checked that $\chi_{\text{us}}(T_k) \leq k$, since the coloring which assigns every vertex of level i the color $k - i$ is a us-coloring of T_k . To see that $\chi_{\text{us}}(T_k) \geq k$, we note that applying [Lemma 3.4](#) on the root r suggests that it can be assigned only colors which are at least k .

4 Planar Graphs

In this section we provide a nearly sharp asymptotic upper-bound on the us-number and the l -vr-number of planar graphs.

4.1 The us-number of Planar Graphs

Theorem 4.1. *For any planar graph G on n vertices, $\chi_{\text{us}}(G) = O(\log n)$.*

Before presenting the proof of [Theorem 4.1](#), we give an informal sketch. Given a planar graph $G = (V, E)$, we begin by partitioning the set of vertices into $t = O(\log n)$ independent sets $V = V_1 \cup \dots \cup V_t$ with the following property: For every $1 \leq i < j \leq t$, any vertex $v \in V_i$ has at most $O(1)$ neighbors in V_j . For each of the sets V_j we shall use a distinct palette of $O(1)$ colors, such that V_i will have smaller colors than V_j for all $1 \leq i < j \leq t$. We shall exhibit a coloring of each set V_j with the property that whenever two vertices $u, v \in V_j$ have a common neighbor in V_i (for $i < j$) then u and v will get distinct colors.

We need the following crucial lemma which provides a coloring with the required property for each of the V_j .

Lemma 4.2. *Let $G = (A \cup B, E)$ be a bipartite planar graph with bipartition A and B . Assume further that every vertex in A has degree at most b for some fixed integer b . Then there exists a coloring $c : B \rightarrow [6b]$ of the vertices in B , such that if $u, v \in B$ have a common neighbor (in A) then $c(u) \neq c(v)$.*

Proof. By a lemma that can be found in, e.g., [\[Deb02\]](#), there exists a plane graph $G' = (B, E')$ with a set of faces F and a bijection $g : A \rightarrow F'$ where $F' \subset F$, that satisfies the following property: For any two vertices $x \in A$, $y \in B$ we have that $\{x, y\} \in E(G)$ if and only if y is incident in G' with the face $f = g(x)$. Our goal is to show that there exists a coloring of the vertices of G' with $6b$ colors, such that all vertices incident on a face $f \in F'$ get distinct colors. We define the auxiliary graph G'' by adding edges to G' . For every $f \in F'$ we place edges between any pair of vertices on f .

It remains to prove that $\chi(G'') \leq 6b$. We do this by showing that G'' is $(6b - 1)$ -degenerate. Note that G'' is obtained from G' by adding cliques to a subset of the faces. Since the degree in G of any vertex in A is at most b , each of the faces in F' has at most b vertices on its boundary. As each clique of size b is contained in a union of $b - 1$ trees², we have that G'' is the union of at most $b - 1$ planar graphs. The same argument holds for any subgraph of G'' , thus, as the average degree of a planar graph is bounded by 6, the average degree in a union of $b - 1$ planar graphs is at most $6(b - 1)$, so G'' is indeed $(6b - 1)$ -degenerate and so $6b$ colorable. \square

Proof of Theorem 4.1. We start by partitioning the vertices into $t = O(\log n)$ independent sets as follows: Let $V' \subset V$ be the subset of vertices with degree less than 12. It is easy to see that $|V'| \geq n/2$. Indeed, by Euler's formula it follows that $|E(G)| \leq 3n - 6$, and by [Proposition 2.3](#),

²If the vertices of the clique are v_1, \dots, v_b , then the tree T_i for $1 \leq i \leq b - 1$, will consist of all edges from v_i to the other vertices.

$|V \setminus V'| < 6n/12$ and hence $|V'| \geq n/2$. Since every planar graph is four-colorable [AH77], there exists a subset $V'' \subset V'$ of size at least $\frac{|V'|}{4} \geq n/8$ which is also independent. Put $V_1 = V''$ and continue recursively on the subgraph $G[V \setminus V_1]$. Since in every iteration we discard at least an $1/8$ fraction of the remaining vertices, the final number of sets is at most $t = \log_{8/7} n + 1$.

Next, we color each of the sets V_i by applying Lemma 4.2 on the bipartite graph induced by $A = \bigcup_{j=1}^{i-1} V_j$ and $B = V_i$ (that is, discarding the internal edges in A and B). Observe that by construction every vertex $y \in A$ has at most 11 neighbors in B , thus there exists a coloring of V_i with 66 colors, such that if $x, z \in B$ have a common neighbor $y \in A$, then x, z get different colors. The total number of colors used for V is at most $66t = O(\log n)$.

To complete the proof of the theorem, we need to show that the coloring c obtained in this manner is a us-coloring. Indeed, the coloring is proper since each V_i is an independent set, and each V_i has different colors than V_j for $i \neq j$. Thus, it is only left to check that if we have a path (x, y, z) for which $c(x) = c(z)$ then $c(y) > c(x)$. Since $c(x) = c(z)$ it must be that $x, z \in V_i$ for some $1 \leq i \leq t$, and by Lemma 4.2 it also holds that $y \in V_j$ for some $j \geq i$ (as otherwise the assertion of the Lemma is violated for this triple). Recall that V_i is an independent set, so in fact $j > i$, and then $c(y) > c(x)$ by construction. \square

4.2 The l -vr-number of Planar Graphs

Here we generalize Theorem 4.1 to the l -vr-number of planar graphs. The main tool is a planar separator theorem due to [Tho04], where the separator consists of three shortest paths.

Definition 4.3. Let $G = (V, E)$ be a graph. An α -separator of G is a set of vertices $U \subseteq V$, such that the number of vertices in any connected component of $G \setminus U$ is at most $\alpha|V|$.

Let T be a rooted tree with root r , and let u be some vertex in T . Denote by $P_T(u)$ the unique simple path in T from r to u . Before introducing the main theorem of this section, we need the following separator theorem of Thorup [Tho04], based on the planar separator of [LT79]:

Theorem 4.4 ([Tho04]). *Let $G = (V, E)$ be a connected planar graph with n vertices and let T be some spanning tree of G with root r . Then there exist three vertices u, v, w such that $P_T(u) \cup P_T(v) \cup P_T(w)$ is a $1/2$ -separator of G .*

The result of this section is the following theorem.

Theorem 4.5. *Let l be some fixed positive integer. Then for any $n > 0$ and any planar graph $G = (V, E)$ with n vertices, $\chi_{l\text{-vr}}(G) \leq \lceil 3(l+1) \log n \rceil$. Furthermore, computing such a coloring can be done in polynomial time.*

Proof. The proof is by induction on n . Fix an arbitrary vertex $r \in V$. Let T be a Breadth-First-Search (BFS) tree of G rooted at r .³ By Theorem 4.4 there exist three vertices $u_0, u_1, u_2 \in V$ such that the set $S = P_T(u_0) \cup P_T(u_1) \cup P_T(u_2)$ is a $1/2$ -separator of G . Every connected component Q in the graph $G \setminus S$ has at most $n/2$ vertices, so by the induction hypothesis $\chi_{l\text{-vr}}(Q) \leq \lceil 3(l+1) \log(n/2) \rceil = \lceil 3(l+1) \log n \rceil - 3(l+1)$. Let $d = \lceil 3(l+1) \log n \rceil - 3(l+1)$, and for each such connected component Q , use the colors $[d]$ to color it inductively (note that we use the same color set for different components). It remains to show that we can complete such a coloring to a legal

³A BFS tree with root r is the tree created by taking the shortest paths from every node u to r (assuming a consistent choice between paths of equal length).

l -vertex ranking of G using additional $3(l+1)$ colors. For $j \in \{0, 1, 2\}$ and a vertex $x \in P_T(u_j)$ with $d_G(x, r) = m$, use color number

$$d + 1 + m \pmod{l+1} + (l+1)j .$$

This implies that vertices on $P_T(u_0)$ will have the colors $d+1, \dots, d+l+1$, vertices on $P_T(u_1)$ will have the colors $d+l+2, \dots, d+2l+2$, and vertices on $P_T(u_2)$ will have the colors $d+2l+3, \dots, d+3l+3$. (If a vertex belong to more than one of $P_T(u_j)$, it can use any one of the colors given to it).

To see that this is indeed an l -vertex ranking of G , let $s, t \in V$ be two vertices with the same color, and let P be a path between s and t of length at most l . Notice that either both s, t are in $V \setminus S$ or both vertices are in S (for otherwise, they cannot get the same color). If the former case is true, then there are two subcases: Either P is entirely contained in $V \setminus S$, thus it lies inside a single connected component of $G \setminus S$, and by the induction hypothesis P contains a vertex with color greater than the color of s and t . The other subcase is that there exist some vertex $x \in P \cap S$, but then the color of x is strictly greater than d , which is the bound on the colors of s and t . If, on the other hand, the latter case holds and $s, t \in S$, then it must be the case that both s and t belong to the same path $P_T(u_j)$ (otherwise they can not have the same color). By a property of a BFS tree, the path between s and t in T is also the shortest path between s and t in G . Yet s and t have the same color, thus by construction the path between them in T is of length at least $l+1$, which is a contradiction.

To see that such a coloring can be obtained in polynomial time, we rely on the fact that finding a BFS tree can be done in linear time. Additionally, by [Tho04], finding the three paths originating from the root can be done in polynomial time. Finally, the recursion occurs at most $\log n$ times, since in each step the number of vertices in each connected component is at most half that of the graph in the previous step. \square

5 Minor-Excluded Graph Families

In this section we extend the results of [Section 4](#) to families of graphs excluding some fixed minor.

5.1 The us-number of Graphs Excluding a Fixed Minor

As our bounds depend only on the cardinality of the minor being excluded, it suffices to consider the family \mathcal{M}_r , consisting of all graphs that exclude K_r as a minor. We begin with a bound on the us-number of such graphs, which generalizes [Theorem 4.1](#).⁴

Theorem 5.1. *For any graph $G \in \mathcal{M}_r$ with n vertices,*

$$\chi_{us}(G) = O((r\sqrt{\log r})^3 \cdot \log n) .$$

The structure of the proof is the same as that of [Theorem 4.1](#) (though the constants will now depend on r). We will need to formulate and prove a lemma similar to [Lemma 4.2](#).

Lemma 5.2. *Let $G = (A \cup B, E)$ be a bipartite graph in \mathcal{M}_r with bipartition A and B . Assume further that every vertex in A has degree at most b for some fixed integer b . Then there exists a coloring $c : B \rightarrow [Cb]$ of the vertices in B , for some $C = O(r\sqrt{\log r})$, such that if u and v are two vertices in B with a common neighbor (in A) then $c(u) \neq c(v)$.*

⁴Recall that planar graphs are in particular K_5 -free.

Proof. W.l.o.g assume that each vertex in A has degree exactly b , and for each vertex $v \in A$, fix some ordering of its b neighbors $\{u_1, \dots, u_b\}$. For each $1 \leq i \leq b$, let $E_i \subseteq E$ be the set of edges between each $v \in A$ and its i -th neighbor. Define a graph G_i , which is obtained from G by contracting all edges in E_i . Observe that we contract exactly one edge for each $v \in A$, thus the vertex set of G_i can be identified with B . Moreover, the graph G_i excludes K_r as a minor, since excluding minors is invariant under edge contraction.

Consider the graph $G' = (B, \cup_{i=1}^b E(G_i))$. We claim that a proper coloring c of G' , will also be a coloring satisfying the assertion of the Lemma. Indeed, if two vertices $x, y \in B$ have a common neighbor $v \in A$, then if x is the i -th neighbor of v , the edge $\{y, v\}$ is not contracted in G_i . Observe that the contraction of $\{v, x\}$ in G_i will have x, y as neighbors, so a proper coloring will provide them with different colors. What remains to be proven, then, is that G' can be properly colored with $O(br\sqrt{\log n})$ colors. By a theorem of Thompson [Tho01], a K_r -minor-free graph has an average degree of $O(r\sqrt{\log r})$. Since G' (and any subgraph of G') is the union of b such graphs, it has an average degree at most $O(br\sqrt{\log r})$, hence it is $O(br\sqrt{\log r})$ -degenerate. As we mentioned earlier, this means that G' can be properly colored with $O(br\sqrt{\log r})$ colors. \square

Proof of Theorem 5.1. We go along similar lines as in the case for us-coloring of planar graphs, create a partition of V to independent sets $V_1 \cup \dots \cup V_t$, such that for every $1 \leq i \leq t$, any $x \in V_i$ has at most $b = b(r)$ neighbors in $\cup_{j=i}^t V_j$.

By [Tho01] the average degree of G is less than $a \cdot r\sqrt{\log r}$ (where a is a universal constant), so let $b = 2a \cdot r\sqrt{\log r}$. Let $V' \subseteq V$ to be the set of vertices whose degree is at most b , and by Proposition 2.3 we get $|V'| \geq n/2$. As $G[V']$ is $(a \cdot r\sqrt{\log r} - 1)$ -degenerate, it can be properly colored using $a \cdot r\sqrt{\log r}$ colors, so there is a color class V'' of size at least $|V''| \geq |V'|/(a \cdot r\sqrt{\log r}) \geq n/b$. Put $V_1 = V''$ and continue recursively on the graph $G[V \setminus V_1]$. At each step we throw out at least $\frac{1}{b}$ fraction of the vertices, which means that $t = O(\log_{1/(1-1/b)} n) = O(b \log n)$.

Now, for each $1 \leq i \leq t$, color V_i with $O(b \cdot r\sqrt{\log r})$ colors, by using Lemma 5.2 on $A = \cup_{j=1}^{i-1} V_j$, $B = V_i$ and the parameter b . From the same considerations as in the proof of Theorem 4.1, we obtain a us-coloring of G of size $O(tb \cdot r\sqrt{\log r}) = O((r\sqrt{\log r})^3 \cdot \log n)$. \square

5.2 The l -vr-number of Graphs Excluding a Fixed Minor

In this section we extend all theorems in this section and Section 4 (albeit, with worse constants), by proving a bound on the l -vr number of graphs excluding a fixed minor. We shall use *path separators* of such graphs due to [AG06], which extend the planar separators of [LT79, Tho04].

Definition 5.3. A graph $G = (V, E)$ on n vertices is *s-path separable* if there exists an integer t and a separator $S \subseteq V$ such that:

1. $S = P_0 \cup P_1 \cup \dots \cup P_t$, where for each $0 \leq i \leq t$, P_i is a collection of shortest paths in the graph $G \setminus (\cup_{0 \leq j < i} P_j)$.
2. $\sum_{i=0}^t |P_i| \leq s$, that is, the total lengths of paths in S is at most s .
3. Each connected component of $G \setminus S$ is s -path separable and has at most $n/2$ vertices;

Theorem 5.4 ([AG06]). *Every H -minor-free graph is s -path separable, for $s = s(H)$, and an s -path separator can be computed in polynomial time.*

Now we are ready to prove the following theorem. The proof is similar to the proof of Theorem 4.5, and we give the full details for completeness.

Theorem 5.5. *Let $G = (V, E)$ be a graph on n vertices that excludes H as a minor, then $\chi_{l\text{-vr}}(G) \leq s(l+1) \log n$, where $s = s(H)$ is a constant that depends only on H . Moreover, such a coloring can be computed in polynomial time.*

Proof. Let S be an s -path separator of G for some $s = s(H)$, as guaranteed by Theorem 5.4. As each connected component of $G \setminus S$ has at most $n/2$ vertices (and of course still excludes H as a minor), we assume inductively that there exists an l -bounded vertex ranking for it using $s(l+1) \log(n/2) = s(l+1) \log n - s(l+1)$ colors. Note that we use the same colors for different components. We will use additional $s(l+1)$ colors for the vertices of S , each of these new colors will be higher than each color used for $V \setminus S$. Each of the paths in S will have its own $l+1$ colors, in such a way that paths in P_j will have higher colors than paths in P_i for all $0 \leq j < i \leq t$. Note that the second property of Definition 5.3 guarantees that we have enough colors. To color a path, simply apply the $l+1$ colors consecutively in a cyclic manner along the path as in Theorem 4.5. If a certain vertex belongs to more than one path, it will keep the highest color assigned to it.

Next we prove the validity of the coloring. To this end, consider a path P of length at most l . If it is the case that P is contained in a connected component of $G \setminus S$, the induction hypothesis guarantees that the coloring is valid, so consider the case that $P \cap S \neq \emptyset$. Let i be the minimal such that $P \cap P_i \neq \emptyset$. By the definition of the coloring, the highest color in P will be one of the colors of P_i . Let Q be the path in P_i such that the highest color in P comes from a vertex of Q . The proof will be concluded once we show that any color in $P \cap Q$ is unique. Seeking contradiction, assume that there are two vertices $u, v \in P \cap Q$ that were assigned the same color. By the first property of Definition 5.3, Q is a shortest path of $G' = G \setminus (\bigcup_{0 \leq j < i} P_j)$, so the distance between u, v in G' must be at least $l+1$. However, the minimality of i suggests that P is also contained in G' , and as P is a path of length at most l , it cannot contain both u, v . \square

6 Degenerate Graphs

In this section we focus on the *us-number* of the family of d -degenerate graphs. This family is a generalization of \mathcal{M}_r , and it could seem plausible that the us-number will be at most polylogarithmic in the number of vertices for degenerate graphs as well. We show that no such bound exists, by showing that for any integer n there exists a 2-degenerate graph G with n vertices satisfying $\chi_{\text{us}}(G) = \Omega(n^{1/3})$. We also prove an upper bound $\chi_{\text{us}}(G) = O(d\sqrt{n})$ for any $G \in \mathbb{G}_d$ with n vertices. These bounds demonstrate the large difference between us-coloring and proper coloring. We also show a large gap between the us-number and vr-number, by proving that there exists a graph $G \in \mathbb{G}_3$ with n vertices such that $\chi_{\text{vr}}(G) = \Omega(n)$.

6.1 Upper Bound on the us-number of Degenerate Graphs

In order to bound the us-number we will need the following Lemma, that provides a bound on the (usual) chromatic number of the squared graph G^2 , for degenerate bounded degree graphs.

Lemma 6.1. *Let $G \in \mathbb{G}_d$ be a graph with maximum degree Δ . Then $\chi(G^2) \leq (2\Delta + 1)d + 1$.*

Proof. Take an ordering of the vertices of G , $V = (v_1, \dots, v_n)$ as in Proposition 2.2. We color the vertices greedily in order. Assume by induction that we have a proper coloring of $G^2[\{v_1, \dots, v_{i-1}\}]$ using at most $(2\Delta + 1)d + 1$ colors. Assign to v_i the lowest possible color c_i . We claim that $c_i \leq (2\Delta + 1)d + 1$. Indeed, since the maximum degree is Δ there can be at most Δ indices $h > i$ such that v_h is a neighbor of v_i . By the property of Proposition 2.2, each v_h can have at most d

neighbors v_j with $j < h$, so there are at most Δd paths of the form (v_i, v_h, v_j) with $j < i < h$. On the other hand, v_i itself has at most d neighbors v_j with $j < i$, each of these has at most Δ neighbors, so there are at most Δd paths of the form (v_i, v_j, v_g) with $j < i, g < i$. We conclude that there are at most $2\Delta d$ vertices among v_1, \dots, v_{i-1} that are of distance 2 from v_i , and at most d of distance 1, therefore v_i has an available color in $[(2\Delta + 1)d + 1]$ such that the coloring of $G^2[\{v_1, \dots, v_i\}]$ is proper. \square

Theorem 6.2. *Let $G \in \mathbb{G}_d$ be a graph with n vertices. Then $\chi_{us}(G) = O(d\sqrt{n})$.*

Proof. Put $U = \{v \in V : d(v) < \sqrt{n}\}$, and let $G[U]$ be the subgraph of G induced by U . Notice that $G[U] \in \mathbb{G}_d$ and $\Delta(G[U]) < \sqrt{n}$, so by Lemma 6.1, $\chi_{us}(G[U]) \leq \chi(G[U]^2) \leq (2\sqrt{n} + 1)d$. Assume that we have such a coloring for U , and it remains to complete the coloring for the large degree vertices $V \setminus U$. For every vertex $v \in V \setminus U$ use a distinct color larger than $(2\sqrt{n} + 1)d$. By Lemma 2.3, $|V \setminus U| \leq \frac{2dn}{\sqrt{n}} = 2d\sqrt{n}$, so in total we have used at most $d(4\sqrt{n} + 1)$ colors. \square

6.2 Lower Bound on the us-number of Degenerate Graphs

Theorem 6.3. *For every n there is a 2-degenerate graph $G = (V, E)$ with n vertices, such that $\chi_{us}(G) > n^{1/3}$.*

Proof. Fix some $k \in \mathbb{N}$, and let G be the graph obtained by taking the complete graph K_k , replicating each edge k times and then subdividing each edge. Notice that G is, indeed, a 2-degenerate graph, since every vertex subdividing an edge has degree 2, and once all these vertices are removed we are left only with isolated vertices. We have $|V| = k + k\binom{k}{2} < k^3$. We will prove that $\chi_{us}(G) \geq k$. Seeking contradiction, assume that $c : V \rightarrow \{1, \dots, m\}$ is a unique-superior coloring of G , with $m < k$. Then, by the pigeon-hole principle, there must be two distinct vertices among the original k vertices of K_k , v_i and v_j , such that $c(v_i) = c(v_j)$. Hence, for every vertex u subdividing an edge between v_i and v_j , $c(u) > c(v_i)$. But notice that it must hold that $c(u) \neq c(u')$ for every two vertices u and u' subdividing two distinct edges between v_i and v_j , as otherwise the path (u, v_i, u') will violate the requirement of a legal us-coloring. Since there are k vertices subdividing two distinct edges between v_i and v_j , we must use at least k distinct colors for them, which yields a contradiction. See Figure 6.2 for an illustration. \square

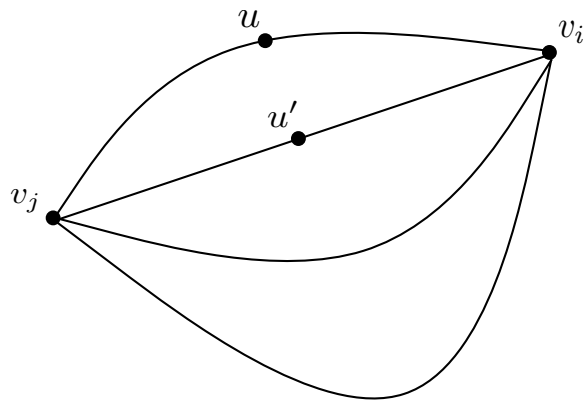


Figure 3: if $c(v_i) = c(v_j)$, then $c(u) \neq c(u')$ for every two subdividing vertices u and u' .

6.3 The vr-number of 3-Regular Graphs

Here we show that the vr-number of a bounded degree graph can be arbitrarily larger than its us-number. In particular, we show that there exists a 3-regular graph G with n vertices such that $\chi_{\text{vr}}(G) = \Omega(n)$. Denote by $\text{pw}(G)$ the *path-width* of G , by $\text{tw}(G)$ the *tree-width* of G , and by $h(G)$ the *vertex expansion* of G . See [BGHK95, GM09] for the definitions of these notions. We need the following known facts:

$$\text{pw}(G) \geq \text{tw}(G) . \quad (1)$$

Lemma 6.4 ([BGHK95]). *For every graph G :*

$$\chi_{\text{vr}}(G) \geq \text{pw}(G) .$$

Lemma 6.5 ([GM09]). *For every graph G :*

$$\text{tw}(G) \geq h(G) \cdot |V(G)|/4 .$$

Remark 6.6. *There exists a constant $h > 0$ and an infinite family \mathcal{F} of 3-regular graphs, such that for all $G \in \mathcal{F}$, $h(G) \geq h$ (see, e.g., [HLW06]).*

Combining (1), Lemma 6.4 and Lemma 6.5, we obtain that the family \mathcal{F} of Remark 6.6 satisfies for all $G \in \mathcal{F}$:

$$\chi_{\text{vr}}(G) \geq \text{pw}(G) \geq \text{tw}(G) \geq h(G) \cdot |V(G)|/4 = \Omega(|V(G)|) .$$

Acknowledgments.

We wish to thank Panagiotis Cheilaris for many helpful discussions.

References

- [ACK⁺04] M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kündgen, and R. Ramamurthi. Coloring with no 2-colored P_4 's. *The Electronic Journal of Combinatorics*, 11, 2004.
- [AG06] I. Abraham and C. Gavoille. Object location using path separators. In *Proceedings of the twenty-fifth annual ACM symposium on Principles of distributed computing*, PODC '06, pages 188–197, New York, NY, USA, 2006. ACM.
- [AH77] K. Appel and W. Haken. Every planar map is four colorable. *Illinois Journal of Mathematics*, 21(3):429–567, Sep. 1977.
- [BDJ⁺98] H. L. Bodlaender, J. S. Deogun, K. Jansen, T. Kloks, D. Kratsch, H. Müller, and Z. Tuza. Rankings of graphs. *SIAM J. Discret. Math.*, 11(1):168–181, February 1998.
- [BGHK95] H. L. Bodlaender, J. R. Gilbert, H. Hafsteinsson, and T. Kloks. Approximating treewidth, pathwidth, frontsize, and shortest elimination tree. *J. Algorithms*, 18(2):238–255, March 1995.
- [Bor79] O.V. Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 25(3):211–236, 1979.

- [Deb02] M. Debowy. Results on planar hypergraphs and on cycle decompositions. Master's thesis, The University of Vermont, 2002.
- [DKKM94] J.S. Deogun, T. Kloks, D. Kratsch, and H. Müller. On vertex ranking for permutation and other graphs. In Patrice Enjalbert, Ernst W. Mayr, and KlausW. Wagner, editors, *STACS 94*, volume 775 of *Lecture Notes in Computer Science*, pages 747–758. Springer Berlin Heidelberg, 1994.
- [DN06] D. Dereniowski and A. Nadolski. Vertex rankings of chordal graphs and weighted trees. *Inf. Process. Lett.*, 98(3):96–100, May 2006.
- [ES11] G. Even and S. Smorodinsky. Hitting sets online and vertex ranking. In *Proceedings of the 19th European conference on Algorithms*, ESA'11, pages 347–357, Berlin, Heidelberg, 2011. Springer-Verlag.
- [GM09] M. Grohe and D. Marx. On tree width, bramble size, and expansion. *J. Comb. Theory Ser. B*, 99(1):218–228, January 2009.
- [Gru73] B. Grunbaum. Acyclic colorings of planar graphs. *Israel Journal of Mathematics*, 14(4):390–408, 1973.
- [HLW06] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc.*, 43(4):439–561, 2006.
- [IRV88] A. V. Iyer, H. D. Ratliff, and G. Vijayan. Optimal node ranking of trees. *Inf. Process. Lett.*, 28(5):225–229, August 1988.
- [KMS95] M. Katchalski, W. McCuaig, and S. Seager. Ordered colourings. *Discrete Mathematics*, 142:141 – 154, 1995.
- [Lei80] C. E. Leiserson. Area-efficient graph layouts. In *Proceedings of the 21st Annual Symposium on Foundations of Computer Science*, SFCS '80, pages 270–281, Washington, DC, USA, 1980. IEEE Computer Society.
- [Liu90] J. W. H. Liu. The role of elimination trees in sparse factorization. *SIAM J. Matrix Anal. Appl.*, 11(1):134–172, January 1990.
- [LT79] R. J. Lipton and R. E. Tarjan. A Separator Theorem for Planar Graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.
- [NOdM03] J. Nešetřil and P. Ossona de Mendez. Colorings and homomorphisms of minor closed classes. In Boris Aronov, Saugata Basu, Jnos Pach, and Micha Sharir, editors, *Discrete and Computational Geometry*, volume 25 of *Algorithms and Combinatorics*, pages 651–664. Springer Berlin Heidelberg, 2003.
- [Sch88] A. A. Schäffer. Optimal vertex ranking of trees in linear time. *Inf. Process. Lett.*, 33:91–99, 1988.
- [Smo12] S. Smorodinsky. Conflict-free coloring and its applications. In I. Barany, K.J. Boroczky, G. Fejes Toth, and J. Pach, editors, *Geometry-Intuitive, Discrete, and Convex*, Bolyai Society Mathematical Studies. Springer, 2012.
- [Tho01] A. Thomason. The extremal function for complete minors. *Journal of Combinatorial Theory, Series B*, 81(2):318 – 338, 2001.

- [Tho04] M. Thorup. Compact oracles for reachability and approximate distances in planar digraphs. *J. ACM*, 51(6):993–1024, 2004.